

■ 利用乘法公式因式分解

對於某些多項式，我們可直接利用乘法公式來做因式分解。

(1) 完全平方公式： $a^2 + 2ab + b^2 = (a+b)^2$

$$a^2 - 2ab + b^2 = (a-b)^2$$

【範例】：因式分解下列各式：(1) $a^2 + 2a + 1$ (2) $x^2 - 10x + 25$

解：(1) $a^2 + 2a + 1$
 $= a^2 + 2 \cdot a \cdot 1 + 1^2$
 $= (a+1)^2$

(2) $x^2 - 10x + 25$
 $= x^2 - 2 \cdot x \cdot 5 + 5^2$
 $= (x-5)^2$

【範例】：因式分解下列各式：(1) $a^2 + 6a + 9$ (2) $4x^2 - 20x + 25$

解：(1) $a^2 + 6a + 9$
 $= a^2 + 2 \cdot a \cdot 3 + 3^2$
 $= (a+3)^2$

(2) $4x^2 - 20x + 25$
 $= (2x)^2 - 2 \cdot (2x) \cdot 5 + 5^2$
 $= (2x-5)^2$

【範例】：因式分解下列各式：(1) $4x^2 - 12xy + 9y^2$ (2) $4x^2 - 20x + 25$

解：(1) $4x^2 - 12xy + 9y^2$
 $= (2x)^2 - 2 \cdot 2x \cdot 3y + (3y)^2$
 $= (2x-3y)^2$

(2) $(x+2y)^2 + 6(x+2y)(y-x) + 9(x-y)^2$
 $= (x+2y)^2 - 2 \cdot (x+2y) \cdot 3(x-y) + [3(x-y)]^2$
 $= [(x+2y) - 3(x-y)]^2$
 $= (-2x+5y)^2$ 或寫成 $(2x-5y)^2$

(2)平方差公式： $a^2 - b^2 = (a+b)(a-b)$

【範例】：因式分解下列各式：(1) $4x^2 - 9$ (2) $x^4 - y^4$

$$\begin{aligned} \text{解} : (1) \quad & 4x^2 - 9 \\ &= (2x)^2 - 3^2 \\ &= (2x-3)(2x+3) \\ (2) \quad & x^4 - y^4 \\ &= (x^2)^2 - (y^2)^2 \\ &= (x^2 - y^2)(x^2 + y^2) \\ &= (x-y)(x+y)(x^2 + y^2) \end{aligned}$$

【範例】：因式分解下列各式：(1) $x^2 - 16$ (2) $4x^2 - 25y$

$$\begin{aligned} \text{解} : (1) \quad & x^2 - 16 \\ &= x^2 - 4^2 \\ &= (x-4)(x+4) \\ (2) \quad & 4x^2 - 25y \\ &= (2x)^2 - (5y)^2 \\ &= (2x-5y)(2x+5y) \end{aligned}$$

【範例】：因式分解下列各式：

$$(1) x^2 - (x+2y)^2 \quad (2) 9 - (a+2)^2 \quad (3) x^2 - y^2 + 2yz - z^2$$

$$\begin{aligned} \text{解} : (1) \quad & x^2 - (x+2y)^2 = [x+(x+2y)][x-(x+2y)] \\ &= (2x+2y)(x-x-2y) \\ &= 2(x+y)(-2y) \\ &= -4y(x+y) \\ (2) \quad & 9 - (a+2)^2 = 3^2 - (a+2)^2 \\ &= [3+(a+2)][3-(a+2)] \\ &= (3+a+2)(3-a-2) \\ &= (a+5)(1-a) \\ (3) \quad & x^2 - y^2 + 2yz - z^2 = x^2 - (y^2 - 2yz + z^2) \\ &= x^2 - (y-z)^2 \\ &= [x+(y-z)][x-(y-z)] \\ &= (x+y-z)(x-y+z) \end{aligned}$$

(3)立方和、立方差公式： $a^3 + b^3 = (a+b)(a^2 - ab + b^2)$
 $a^3 - b^3 = (a-b)(a^2 + ab + b^2)$

【範例】：因式分解下列各式：(1) $x^3 - 1$ (2) $a^3 + 8b^3$ (3) $x^6 - 1$

解：

$$\begin{aligned} (1) \quad x^3 - 1 &= x^3 - 1^3 \\ &= (x-1)(x^2 + x + 1^2) \end{aligned} \qquad \begin{aligned} (2) \quad a^3 + 8b^3 &= a^3 + (2b)^3 \\ &= (a+2b)[a^2 - a \cdot 2b + (2b)^2] \\ &= (a+2b)(a^2 - 2ab + 4b^2) \end{aligned}$$

$$\begin{aligned} (3) \quad x^6 - 1 &= (x^3)^2 - 1^2 \\ &= (x^3 + 1)(x^3 - 1) \\ &= (x+1)(x^2 - x + 1)(x-1)(x^2 + x + 1) \end{aligned}$$

【範例】：因式分解下列各式：(1) $8x^3 + 27$ (2) $\frac{1}{27} - 8a^3$ (3) $x^6 - y^6$

解：

$$\begin{aligned} (1) \quad 8x^3 + 27 &= (2x)^3 + 3^3 \\ &= (2x+3)[(2x)^2 - 2x \times 3 + 3^2] \\ &= (2x+3)(4x^2 - 6x + 9) \end{aligned}$$

$$\begin{aligned} (2) \quad \frac{1}{27} - 8a^3 &= \left(\frac{1}{3}\right)^3 - (2a)^3 \\ &= \left(\frac{1}{3} - 2a\right) \left[\left(\frac{1}{3}\right)^2 + \frac{1}{3} \times 2a + (2a)^2\right] \\ &= \left(\frac{1}{3} - 2a\right) \left(\frac{1}{9} + \frac{2}{3}a + 4a^2\right) \end{aligned}$$

$$\begin{aligned} (3) \quad x^6 - y^6 &= (x^3)^2 - (y^3)^2 \\ &= (x^3 + y^3)(x^3 - y^3) \\ &= (x+y)(x^2 - xy + y^2)(x-y)(x^2 + xy + y^2) \end{aligned}$$

【範例】：因式分解下列各式：(1) $8x^3 + 27$ (2) $\frac{1}{27} - 8a^3$ (3) $x^6 - y^6$

解：

$$\begin{aligned}(1) \quad & 8x^3 + 27 \\ &= (2x)^3 + 3^3 \\ &= (2x + 3)[(2x)^2 - 2x \times 3 + 3^2] \\ &= (2x + 3)(4x^2 - 6x + 9)\end{aligned}$$

$$\begin{aligned}(2) \quad & \frac{1}{27} - 8a^3 = \left(\frac{1}{3}\right)^3 - (2a)^3 \\ &= \left(\frac{1}{3} - 2a\right)\left[\left(\frac{1}{3}\right)^2 + \frac{1}{3} \times 2a + (2a)^2\right] \\ &= \left(\frac{1}{3} - 2a\right)\left(\frac{1}{9} + \frac{2}{3}a + 4a^2\right)\end{aligned}$$

$$\begin{aligned}(3) \quad & x^6 - y^6 = (x^3)^2 - (y^3)^2 \\ &= (x^3 + y^3)(x^3 - y^3) \\ &= (x + y)(x^2 - xy + y^2)(x - y)(x^2 + xy + y^2)\end{aligned}$$



小 試 身 手

【例題一】《平方差公式》

因式分解下列多項式：

$$(1) x^2 - 81 \quad (2) 36 - 25x^2$$

$$\text{解：}(1) x^2 - 81 = (x + 9)(x - 9)$$

$$(2) 36 - 25x^2 = (6 + 5x)(6 - 5x)$$

【練習一】《平方差公式》

因式分解下列多項式：

$$(1) 16x^2 - 49 \quad (2) 4a^2 - 9b^2$$

$$\text{解：}(1) 16x^2 - 49 = (4x + 7)(4x - 7)$$

$$(2) 4a^2 - 9b^2 = (2a + 3b)(2a - 3b)$$

【例題二】《利用平方差公式求值》

求出下列各式的值：

$$(1) 198^2 - 98^2 \quad (2) 40.8 \times 39.2$$

$$\begin{aligned} \text{解：}(1) 198^2 - 98^2 \\ &= (198 + 98)(198 - 98) \\ &= 296 \times 100 = 29600 \end{aligned}$$

$$\begin{aligned} (2) 40.8 \times 39.2 &= (40 + 0.8)(40 - 0.8) \\ &= 40^2 - 0.8^2 = 1600 - 0.64 \\ &= 1599.36 \end{aligned}$$

【練習二】《利用平方差公式求值》

求出下列各式的值：

$$(1) 9 - 997^2 \quad (2) 59\frac{1}{3} \times 60\frac{2}{3}$$

$$\begin{aligned} \text{解：}(1) 9 - 997^2 &= (3 + 997)(3 - 997) \\ &= 1000 \times (-994) = -994000 \end{aligned}$$

$$\begin{aligned} (2) 59\frac{1}{3} \times 60\frac{2}{3} &= (60 - \frac{2}{3})(60 + \frac{2}{3}) \\ &= 60^2 - (\frac{2}{3})^2 = 3600 - \frac{4}{9} \\ &= 3599\frac{5}{9} \end{aligned}$$

【例題三】《利用平方差公式因式分解》

因式分解下列多項式：

(1) $9x^2 - 1$

(2) $(3x - 5)^2 - (x + 1)^2$

(3) $2(x^2 - y^2) - (x - y)$

解：(1) $9x^2 - 1 = (3x + 1)(3x - 1)$

$$\begin{aligned} (2) \quad & (3x - 5)^2 - (x + 1)^2 \\ & = (4x - 4)(2x - 6) \\ & = 8(x - 1)(x - 3) \end{aligned}$$

$$\begin{aligned} (3) \quad & 2(x^2 - y^2) - (x - y) \\ & = 2(x + y)(x - y) - (x - y) \\ & = (x - y) [2(x + y) - 1] \\ & = (x - y)(2x + 2y - 1) \end{aligned}$$

【例題四】《利用完全平方公式因式分解》

因式分解下列多項式：

(1) $2xy^2 - y^3 - x^2y$

(2) $a^2 - b^2 + 2bc - c^2$

解：(1) $2xy^2 - y^3 - x^2y$

$= -y(x^2 - 2xy + y^2)$

$= -y(x - y)^2$

$= -y(x - y)^2$

$$\begin{aligned} (2) \quad & a^2 - b^2 + 2bc - c^2 \\ & = a^2 - (b^2 - 2bc + c^2) \\ & = a^2 - (b - c)^2 \\ & = [a + (b - c)][a - (b - c)] \\ & = (a + b - c)(a - b + c) \end{aligned}$$

【練習三】《利用平方差公式因式分解》

因式分解下列多項式：

(1) $x^3 - 3x^2 - 4x + 12$

(2) $x(x + z) - y(y - z)$

(3) $x^2(y - z) + y^2(z - x)$

解：(1) $x^3 - 3x^2 - 4x + 12$

$= x^2(x - 3) - 4(x - 3)$

$= (x - 3)(x^2 - 4)$

$= (x - 3)(x + 2)(x - 2)$

$$\begin{aligned} (2) \quad & x(x + z) - y(y - z) \\ & = x^2 + xz - y^2 + yz \\ & = (x^2 - y^2) + (xz + yz) \\ & = (x + y)(x - y) + z(x + y) \\ & = (x + y)(x - y + z) \end{aligned}$$

$$\begin{aligned} (3) \quad & x^2(y - z) + y^2(z - x) \\ & = x^2y - x^2z + y^2z - xy^2 \\ & = (x^2y - xy^2) - (x^2z - y^2z) \\ & = xy(x - y) - z(x^2 - y^2) \\ & = xy(x - y) - z(x + y)(x - y) \\ & = (x - y)(xy - xz - yz) \end{aligned}$$

【練習四】《利用完全平方公式因式分解》

因式分解下列多項式：

(1) $x^2 - xy - 2x + y + 1$

(2) $4x^2y^2 + 2xy - x^2 - y^2$

解：(1) $x^2 - xy - 2x + y + 1$

$= (x^2 - 2x + 1) - y(x - 1)$

$= (x - 1)^2 - y(x - 1)$

$= (x - 1)(x - y - 1)$

$$\begin{aligned} (2) \quad & 4x^2y^2 + 2xy - x^2 - y^2 \\ & = (2xy)^2 - (x^2 - 2xy + y^2) \\ & = (2xy)^2 - (x - y)^2 \\ & = (2xy + x - y)(2xy - x + y) \end{aligned}$$

【例題五】

分解 $x^2 - y^2 + 2yz - z^2$

$$\begin{aligned} \text{解：} & x^2 - (y^2 - 2yz + z^2) \\ & = x^2 - (y - z)^2 \\ & = (x + y - z)(x - y + z) \end{aligned}$$

【練習五】

分解 $a^4 - 2a^2b^2 + b^4$

$$\begin{aligned} \text{解：} & a^4 - 2a^2b^2 + b^4 \\ & = (a^2)^2 - 2(a^2)(b^2) + (b^2)^2 \\ & = (a^2 - b^2)^2 \\ & = [(a+b)(a-b)]^2 \\ & = (a+b)^2(a-b)^2 \end{aligned}$$

【例題六】

若 $2a^2 + b^2 + c^2 - 12a - 8b - 10c + 59 = 0$ ，求 $2a + 3b + 4c$ 為多少？

$$\begin{aligned} \text{解：} & 2a^2 + b^2 + c^2 - 12a - 8b - 10c + 59 = 0 \\ & 2(a^2 - 6a) + (b^2 - 8b) + (c^2 - 10c) + 59 = 0 \\ \therefore & 2(a^2 - 6a + 9) + (b^2 - 8b + 16) + (c^2 - 10c + 25) = 0 \\ \therefore & 2(a-3)^2 + (b-4)^2 + (c-5)^2 = 0 \quad \therefore a=3, b=4, c=5 \\ \therefore & 2a + 3b + 4c = 6 + 12 + 20 = 38 \end{aligned}$$

【練習六】

設 $a^2 + b^2 + c^2 = 2$ ， $ab + bc + ac = 1$ ，求 $a + b + c$ 之值。

$$\begin{aligned} \text{解：} & \therefore (a+b+c)^2 = a^2 + b^2 + c^2 + 2(ab+bc+ac) = 2 + 2 = 4 \\ & \therefore a+b+c = \pm 2 \end{aligned}$$

【例題七】《利用乘法公式求值》

若 $a - b = 4$ ， $ab = -3$ ，求：

(1) $a^2 + b^2$ 之值 (2) $a + b$ 之值 (3) $a^4 + b^4$ 之值 (4) $a^4 - b^4$ 之值。

$$\text{解：(1) } a^2 + b^2 = (a-b)^2 + 2ab = 16 - 6 = 10$$

$$\begin{aligned} \text{(2) } & \therefore (a+b)^2 = a^2 + 2ab + b^2 = 10 - 6 = 4 \\ & \therefore a+b = \pm 2 \end{aligned}$$

$$\text{(3) } a^4 + b^4 = (a^2 + b^2)^2 - 2a^2b^2 = 100 - 2 \times 9 = 82$$

$$\begin{aligned} \text{(4) } & a^4 - b^4 = (a^2)^2 - (b^2)^2 = (a^2 + b^2)(a^2 - b^2) \\ & = (a^2 + b^2)(a+b)(a-b) = 10 \times (\pm 2) \times 4 = \pm 80 \end{aligned}$$

【練習七】《利用乘法公式求值》

若 $a-b=8$, $ab=20$, 求：

(1) a^2+b^2 之值 (2) $a+b$ 之值 (3) a^4+b^4 之值 (4) a^4-b^4 之值。

解：(1) $a^2+b^2=(a-b)^2+2ab=64+40=104$

(2) $\because (a+b)^2=a^2+2ab+b^2=104+40=144$

$\therefore a+b=\pm 12$

(3) $a^4+b^4=(a^2+b^2)^2-2a^2b^2=104^2-2\times 400=10816-800=10016$

(4) $a^4-b^4=(a^2)^2-(b^2)^2=(a^2+b^2)(a^2-b^2)$
 $= (a^2+b^2)(a+b)(a-b)=104\times(\pm 12)\times 8=\pm 9984$

【例題八】《利用乘法公式求值》

(1) 12^3+8^3

(2) 7^3-5^3

解：(1) $12^3+8^3=(12+8)(12^2-12\times 8+8^2)$
 $=20\times(144-96+64)=20\times 112=2240$

(2) $7^3-5^3=(7-5)(7^2+7\times 5+5^2)$
 $=2\times(49+35+25)=2\times 109=218$

【練習八】《利用乘法公式求值》

(1) 7^3+3^3

(2) 6^3-14^3

解：(1) $7^3+3^3=(7+3)(7^2-7\times 3+3^2)$
 $=10\times(49-21+9)=10\times 37=370$

(2) $6^3-14^3=(6-14)(6^2+6\times 14+14^2)$
 $=-8\times(36+84+196)=-8\times 316=-2528$